

Geometric Scattering for Schrödinger Operators with Asymptotically Homogeneous Potentials of Order Zero

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Abstract

In this paper we consider Schrödinger operators with potentials of order zero on asymptotically conic manifolds. We prove the existence and the completeness of the wave operators with a naturally defined free Hamiltonian.

1 Introduction

Let M be a smooth n -dimensional manifold. We suppose that M is a union of a relatively compact part M_c and a non-compact part M_∞ . We assume M_∞ is diffeomorphic to $\mathbb{R}_+ \times \partial M$ where ∂M is a smooth compact manifold. By fixing an identification map, we identify a point x in M_∞ with (r, θ) in $\mathbb{R}_+ \times \partial M$. We also assume $M_c \cap M_\infty$ is contained in $(0, 1) \times \partial M$ under the above identification. We define a reference manifold by $M_f = \mathbb{R} \times \partial M$.

We fix a local coordinate system $(U_\alpha, \varphi_\alpha, (\theta_j)_{j=1}^{n-1})$ for ∂M and define a local coordinate system for M_∞ by $(\mathbb{R}_+ \times U_\alpha, I \otimes \varphi_\alpha, (r, (\theta_j)_{j=1}^{n-1}))$.

Let $H(\theta)$ be a positive smooth density on ∂M , and $G(x)$ be a positive smooth density on M such that $G(x) = r^{n-1}H(\theta)$ in $\tilde{M}_\infty = (1, \infty) \times \partial M$. We set \mathcal{H} by $\mathcal{H} = L^2(M, G(x)dx)$ and \mathcal{H}_f by $\mathcal{H}_f = L^2(M_f, H(\theta)d\text{rd}\theta)$.

We fix a smooth cut-off function $j \in C^\infty(\mathbb{R})$ such that $j(r) = 1$ if $1 \leq |r|$ and $j(r) = 0$ if $|r| \leq \frac{1}{2}$. We define $J : \mathcal{H}_f \rightarrow \mathcal{H}$ by

$$(J\varphi)(x) = \begin{cases} r^{-(n-1)/2} j(r) \varphi(r, \theta) & \text{if } x = (r, \theta) \in M_\infty \\ 0 & \text{if } x \notin M_\infty, \end{cases}$$

for $\varphi \in \mathcal{H}_f$.

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Let P be an elliptic second order differential operator on M . We assume P is bounded from below and P is symmetric with $\mathcal{D}(P) = C_c^\infty(M)$. We also assume P is of the form,

$$P = -\frac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta/r \end{pmatrix} + V \quad \text{on } \tilde{M}_\infty,$$

where $\{a_i\}_{i=1}^3$ are real-valued and positive definite smooth tensors on M , and V is a smooth function on M . We assume $\{a_i\}_{i=1}^3$ and V satisfy Assumption A.

Assumption A. Let $\mu_1, \mu_2, \mu_3 > 1$

- For any $\ell \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n-1}$, there exists $C_{\ell\alpha} > 0$ such that

$$\begin{aligned} |\partial_r^\ell \partial_\theta^\alpha (a_1(r, \theta) - 1)| &\leq C_{\ell\alpha} r^{-\mu_1 - \ell}, \\ |\partial_r^\ell \partial_\theta^\alpha a_2(r, \theta)| &\leq C_{\ell\alpha} r^{-\mu_2 - \ell}, \\ |\partial_r^\ell \partial_\theta^\alpha (a_3(r, \theta) - h(\theta))| &\leq C_{\ell\alpha} r^{-\mu_3 - \ell} \end{aligned}$$

on \tilde{M}_∞ where $h(\theta) = \{h^{j,k}(\theta)\}$ is a positive symmetric smooth (2,0)-tensor on ∂M .

- V is real-valued and has decomposition

$$V(r, \theta) = \tilde{V}(\theta) + V_s(r, \theta) \quad \text{if } (r, \theta) \in \tilde{M}_\infty,$$

where $V_s \in C^\infty(M)$ is real-valued and short-range, i.e., there exists $\mu_4 > 1$ such that for any $\ell \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n-1}$,

$$|\partial_r^\ell \partial_\theta^\alpha V_s(r, \theta)| \leq C_{\ell\alpha} r^{-\mu_4 - \ell} \quad \text{if } (r, \theta) \in \tilde{M}_\infty$$

with a constant $C_{\ell\alpha} > 0$. We also assume that $\tilde{V} \in C^\infty(\partial M)$ and that the set of critical value $\text{Cv}(\tilde{V}) = \{\lambda \in \mathbb{R} \mid \lambda = \tilde{V}(\theta), \partial_\theta \tilde{V}(\theta) = 0\}$ is finite.

Remark. Assumption A is a slightly more strict assumption for the perturbation of a_2 and a_3 than that of [7]. This is necessary since we employ the smooth perturbation theory to prove the completeness of the wave operators.

We fix a smooth function $\tilde{j} \in C^\infty(\mathbb{R})$ such that \tilde{j} is strictly positive and $\tilde{j}(r) = 1$ if $|r| \leq \frac{1}{2}$ and $\tilde{j}(r) = \frac{1}{r^2}$ if $|r| \geq 1$. We define the reference operator P_f by

$$P_f = -\frac{1}{2} \frac{\partial^2}{\partial r^2} - \frac{\tilde{j}(r)}{2} \sum_{j,k} H(\theta)^{-1} \partial_{\theta_j} H(\theta) h^{jk}(\theta) \partial_{\theta_k} + \tilde{V}(\theta) \quad \text{on } M_f,$$

We note that P_f is elliptic from the assumption for \tilde{j} .

Theorem 1. (1) The wave operators $W_{\pm} = \text{s-lim}_{t \rightarrow \pm\infty} e^{itP} J e^{-itP_f} P_{ac}(P_f)$ exist, where $P_{ac}(\cdot)$ is the orthogonal projection onto the absolutely continuous subspace.

(2) The wave operators W_{\pm} are complete, i.e., $\text{Ran } W_{\pm} = \mathcal{H}_{ac}(P)$, where $\mathcal{H}_{ac}(\cdot)$ denotes the absolutely continuous subspace.

The study of the Schrödinger operators with potentials of order 0 was initiated by Herbst in [4], who characterized unitary equivalence of such operators and studied asymptotic behavior of time evolution in the Euclidean space case. He also investigated classical mechanics of Hamiltonian flow with potentials of order 0. In [6], Herbst and Skibsted compared Schrödinger operators with potentials of order 0 with Laplacian and proved the existence and asymptotic completeness of wave operators in high and low energy.

Agmon, Cruz, and Herbst considered a class of Schrödinger operators including Schrödinger operators with potentials of order 0. They showed Schrödinger operators with potentials of order 0 can be diagonalized in high energy by solving the eikonal equation in [1](see also Saitō [10]). They also showed that Schrödinger operators with potentials of order 0 satisfy the Mourre estimate with the modified conjugate operators.

One interesting property of the Schrödinger operators with potentials of order 0 is so called “localization of the solution in direction” which is proved by Herbst and Skibsted in [6]. Let P be a Schrödinger operator with potentials of order 0 on \mathbb{R}^n and the variable θ be in S^{n-1} . We define $\mathcal{H}_{\theta} \subset L^2(\mathbb{R}^n)$ by $\{\varphi \in L^2(\mathbb{R}^n) \mid (\frac{x}{|x|} - \theta)e^{-itP}\varphi \rightarrow 0 \text{ as } t \rightarrow \infty\}$. Then it is shown that there exists $\{\theta_m\}_{m=1}^M$ such that $L^2(\mathbb{R}^n) = \oplus_{m=1}^M \mathcal{H}_{\theta_m}$.

On the other hand, scattering theory for the Schrödinger operators with potentials of order 0 on the asymptotically conic manifolds is studied by Hassel, Melrose and Vasy in [2] and [3] under the setting of [9]. They showed the asymptotic completeness and the localization of the solution in direction.

There are several formulation of geometric scattering and we employ the formulation of [7]. A key idea of the formulation in [7] is that they compare Schrödinger operators on the asymptotically conic manifolds with the simpler Schrödinger operators on the asymptotically tubic manifolds. In [8], it is proved that scattering matrix of Schrödinger operators with short-range potential can be written as a Fourier integral operator associated with the asymptotic classical flow in the phase space.

Scattering theory we propose is not a generalization of the results for the Euclidean case since we compare asymptotically the operators with the same leading asymptotic terms on different manifolds. We note that spectral properties presented in Section 2 are straight forward generalizations of previous results in [1] and [4]. Our model is different from that of [2] and [3]. They only treat the case when the potential is the Morse function and they employ more strict assumption for the perturbation of P .

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2 Spectral properties of P and P_f

2.1 Preparation for the proof of the Mourre estimate.

From Corollary 4 of [7], we learn P is self-adjoint. We can prove that P_f is self-adjoint similarly to the proof of Proposition 3 in [7] since P_f is elliptic.

We define a smooth vector field X_λ on M for $\lambda > 0$ by

$$X_\lambda = \begin{cases} j(r) r \frac{\partial}{\partial r} - \sum_{j,k} \frac{1}{\lambda} j(r) (\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta) \partial_{\theta_k} & \text{on } M_\infty \\ 0 & \text{on } M_c. \end{cases}$$

Then X_λ generates flow $\exp(-tX_\lambda)$ for $t \in \mathbb{R}$. We define the unitary group $U_\lambda(t)$ on \mathcal{H} as follows:

$$U_\lambda(t)\varphi(x) = \Phi(t, x)\varphi(\exp(-tX_\lambda)x),$$

where $\Phi(t, x)$ is a positive and smooth weight function to make $U_\lambda(t)$ unitary. Since $U_\lambda(t)$ is unitary, it can be written as $U_\lambda(t) = e^{itA_\lambda}$ where A_λ is a self-adjoint operator. From direct calculation, we can write A_λ as follows on $C_c^\infty(M)$:

$$\begin{aligned} A_\lambda &= \frac{1}{2i} G(r, \theta)^{-\frac{1}{2}} \left(j(r) r \frac{\partial}{\partial r} + \frac{\partial}{\partial r} j(r) r - \sum_{j,k} \frac{1}{\lambda} j(r) (\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta) \partial_{\theta_k} \right. \\ &\quad \left. - \sum_{j,k} \frac{1}{\lambda} \partial_{\theta_j} h^{jk}(\theta) (\partial_{\theta_k} \tilde{V}(\theta)) j(r) \right) G(r, \theta)^{\frac{1}{2}} \\ &= \frac{1}{2i} \left(j(r) r \frac{\partial}{\partial r} + \frac{\partial}{\partial r} j(r) r + (n-1)j(r) - \sum_{j,k} \frac{1}{\lambda} j(r) (\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta) \partial_{\theta_k} \right. \\ &\quad \left. - \sum_{j,k} \frac{1}{\lambda} \partial_{\theta_j} h^{jk}(\theta) (\partial_{\theta_k} \tilde{V}(\theta)) j(r) - \sum_{j,k} \frac{1}{\lambda} (\partial_{\theta_j} H(\theta)) h^{jk}(\theta) (\partial_{\theta_k} \tilde{V}(\theta)) j(r) \right). \end{aligned}$$

By the similar argument, we define a self-adjoint operator A_f on \mathcal{H}_f as follows:

$$\begin{aligned}
A_f &= \frac{1}{2i} H(\theta)^{-\frac{1}{2}} \left(j(r) r \frac{\partial}{\partial r} + \frac{\partial}{\partial r} j(r) r - \sum_{j,k} \frac{1}{\lambda} j(r) (\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta) \partial_{\theta_k} \right. \\
&\quad \left. - \sum_{j,k} \frac{1}{\lambda} \partial_{\theta_j} h^{jk}(\theta) (\partial_{\theta_k} \tilde{V}(\theta)) j(r) \right) H(\theta)^{\frac{1}{2}} \\
&= \frac{1}{2i} \left(j(r) r \frac{\partial}{\partial r} + \frac{\partial}{\partial r} j(r) r - \sum_{j,k} \frac{1}{\lambda} j(r) (\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta) \partial_{\theta_k} \right. \\
&\quad \left. - \sum_{j,k} \frac{1}{\lambda} \partial_{\theta_j} h^{jk}(\theta) (\partial_{\theta_k} \tilde{V}(\theta)) j(r) - \sum_{j,k} \frac{1}{\lambda} (\partial_{\theta_j} H(\theta)) h^{jk}(\theta) (\partial_{\theta_k} \tilde{V}(\theta)) j(r) \right).
\end{aligned}$$

The main part of this section is to prove the following theorem.

Theorem 2. *Let P satisfies Assumption A and we assume λ is sufficiently large. Then*

- (1) *The point spectrum of P is discrete in \mathbb{R} .*
- (2) *Let $I \Subset \mathbb{R} \setminus (\sigma_{pp}(P) \cup \text{Cv}(\tilde{V}))$ and let $s > 1/2$. Then*

$$\sup_{\text{Re } z \in I, \text{Im } z \neq 0} \|\langle A_\lambda \rangle^{-s} (P - z)^{-1} \langle A_\lambda \rangle^{-s}\| < \infty.$$

In particular, $\sigma_{sc}(P) = \emptyset$ and $\langle A_\lambda \rangle^{-s} (P - z)^{-1} \langle A_\lambda \rangle^{-s}$ is Hölder continuous in z if $\text{Re } z \in I$ and $\text{Im } z \neq 0$.

- (3) *Let I be as above, and let $s > 1/2$. Then*

$$\sup_{\text{Re } z \in I, \text{Im } z \neq 0} \|\langle r \rangle^{-s} (P - z)^{-1} \langle r \rangle^{-s}\| < \infty.$$

In particular, $\langle r \rangle^{-s} (P - z)^{-1} \langle r \rangle^{-s}$ is Hölder continuous in z if $\text{Re } z \in I$ and $\text{Im } z \neq 0$. Thus

$$\langle r \rangle^{-s} (P - E \pm i0)^{-1} \langle r \rangle^{-s} = \lim_{\varepsilon \rightarrow +0} \langle r \rangle^{-s} (P - E \pm i\varepsilon)^{-1} \langle r \rangle^{-s}$$

converge uniformly in $E \in I$.

We can prove the same theorem for P_f , i.e.,

Proposition 3. *Theorem 2 holds for P_f with A_λ replaced by A_f .*

From the Mourre theorem in [11], it suffices to prove following Theorem.

Theorem 4. *Let P and A_λ be as above and we assume λ is sufficiently large. For each interval $I \Subset \mathbb{R} \setminus (\sigma_{pp}(P) \cup \text{Cv}(\tilde{V}))$, there exist $\alpha > 0$ and a compact operator K such that*

$$\chi_I(P) i[P, A_\lambda] \chi_I(P) \geq \alpha \chi_I(P) + K,$$

where χ_I is the indicator function of I .

Let m be real number. We consider a symbol class $S^m(T^*M_\infty)$ such that $a \in S^m(T^*M_\infty)$ means $a \in C^\infty(T^*M_\infty) = C^\infty(T^*\mathbb{R}_+ \times T^*\partial M)$ and for each multi-index $\alpha, \beta, \gamma, \delta$ there exists $C_{\alpha, \beta, \gamma, \delta} > 0$ with

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta a(r, \rho, \theta, \omega)| \leq C_{\alpha, \beta, \gamma, \delta} \langle r \rangle^{m-|\alpha|} \langle \rho \rangle^{m-|\gamma|} \langle \omega \rangle^{m-|\delta|}.$$

We quantize $a \in S^m(T^*M_\infty)$ by following [8]. Let $\{\varphi_\alpha, U_\alpha\}$ be a local coordinate system of ∂M and $\{\chi_\alpha^2\}$ be a partition of unity on ∂M compatible with our coordinate system, i.e., $\chi_\alpha \in C_0^\infty(U_\alpha)$ and $\sum_\alpha \chi_\alpha(\theta)^2 \equiv 1$ on ∂M . We also denote $\tilde{\chi}_\alpha(r, \theta) = \chi_\alpha(\theta)j(r) \in C^\infty(M_\infty)$. Let $u \in C_0^\infty(T^*M)$. We denote by $a_{(\alpha)}$ and $G_{(\alpha)}$ the representation of a and G in the local coordinate $(1 \otimes \varphi_\alpha, \mathbb{R} \times U_\alpha)$, respectively. We quantize a by

$$Op^W(a)u = \sum_\alpha \tilde{\chi}_\alpha G_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) G_{(\alpha)}^{1/2} \tilde{\chi}_\alpha u$$

From general theory of the pseudodifferential operators in [12], we directly obtain following Proposition:

Proposition 5. *Let $a \in S^m(T^*M_\infty)$, then $Op^W(a)$ is a compact operator on \mathcal{H} if $m < 0$.*

From Weyl calculus in [8] and [12], we can represent A_λ and P as a pseudodifferential operator.

Proposition 6. *Let*

$$p(r, \rho, \theta, \omega) = \frac{1}{2}(\rho, \omega/r) \begin{pmatrix} a_1(r, \theta) & a_2(r, \theta) \\ {}^t a_2(r, \theta) & a_3(r, \theta) \end{pmatrix} \begin{pmatrix} \rho \\ \omega/r \end{pmatrix} + V(r, \theta) \text{ and}$$

$$a_\lambda(r, \theta) = j(r) r \rho - \sum_{j,k} \frac{1}{\lambda} j(r) (\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta) \omega_k.$$

Then we obtain following formula:

$$Op^W(p) = P + E_1 \text{ and } Op^W(a_\lambda) = A_\lambda + E_2,$$

where E_1, E_2 satisfy the condition $\chi_I(P)E_i\chi_I(P)$ are compact for $i = 1, 2$.

2.2 Proof of the Mourre estimate.

The proof of Theorem 4 is based on that of Euclidean case in Appendix C of [1] and we only prove Theorem 2 since the proof of Proposition 3 is exactly the same with that of Theorem 2.

We have to prepare lemma to prove Theorem 4. Let $P_0 = P - j(2r)\tilde{V}(\theta)$.

Lemma 7. *Let P, λ , and I satisfy the assumption for Theorem 4. Then,*

$$i[P, A_\lambda] \geq P_0 + \frac{1}{\lambda} \sum_{j,k} j(r) (\partial_{\theta_j} \tilde{V}) h^{jk}(\theta) (\partial_{\theta_k} \tilde{V}) + E,$$

where E satisfies the condition $\chi_I(P)E\chi_I(P)$ is compact.

Proof. From, Proposition 6 and locally compactness of P , we obtain

$$i[P, A_\lambda] = i[Op^W(p) - V, Op^W(a_\lambda)] + \frac{j(r)}{\lambda} \sum_{j,k} (\partial_{\theta_j} \tilde{V}) h^{jk}(\theta) (\partial_{\theta_k} \tilde{V}) + E_3,$$

where E_3 satisfies the condition $\chi_I(P)E_3\chi_I(P)$ is compact.

From the formula for the commutator of the pseudodifferential operators in [12], we see

$$[Op^W(p), Op^W(a_\lambda)] = Op^W(\{p, a_\lambda\}) + E_4,$$

where E_4 satisfies the condition $\chi_I(P)E_4\chi_I(P)$ is compact and $\{\cdot, \cdot\}$ denotes the Poisson bracket.

Let $x = (r, \theta)$ and $\zeta = (\rho, \omega)$, we obtain the followings:

$$\begin{aligned} 2\partial_x(p - V) &= (\rho, \omega) \left\{ \partial_x \begin{pmatrix} a_1(r, \theta) & a_2(r, \theta)/r \\ {}^t a_2(r, \theta)/r & a_3(r, \theta)/r^2 \end{pmatrix} \right\} \begin{pmatrix} \rho \\ \omega \end{pmatrix}, \\ 2\partial_\zeta(p - V) &= 2 \begin{pmatrix} a_1(r, \theta) & a_2(r, \theta) \\ {}^t a_2(r, \theta) & a_3(r, \theta)/r \end{pmatrix} \begin{pmatrix} \rho \\ \omega/r \end{pmatrix}, \\ 2\partial_x a_\lambda &= \left(\{\partial_r(j(r)r)\} \rho - \sum_{j,k} \frac{(\partial_r j(r))}{\lambda} (\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta) \omega_k, \right. \\ &\quad \left. \frac{j(r)}{\lambda} \sum_{j,k} {}^t \partial_\theta \{(\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta)\} \omega_k \right), \\ 2\partial_\zeta a_\lambda &= \left(j(r)r, \left(- \sum_j \frac{1}{\lambda} j(r) (\partial_{\theta_j} \tilde{V}(\theta)) h^{jk}(\theta) \right)_k \right), \\ \{p - V, a_\lambda\} &= \partial_x a_\lambda \partial_\zeta(p - V) - \partial_\zeta a_\lambda \partial_x(p - V) \\ &= (\rho, \omega/r) \begin{pmatrix} a_1(r, \theta) & a_2(r, \theta) \\ {}^t a_2(r, \theta) & a_3(r, \theta) \end{pmatrix} \begin{pmatrix} \rho \\ \omega/r \end{pmatrix} - \frac{1}{\lambda} (\rho, \omega/r) B \begin{pmatrix} \rho \\ \omega/r \end{pmatrix} + E_5, \end{aligned}$$

where $B = \mathcal{O}(1)$ and E_5 satisfies the condition $\chi_I(P)E_5\chi_I(P)$ is compact. By taking λ sufficiently large, we obtain

$$\{p - V, a_\lambda\} > P_0 + E,$$

where E satisfies the condition $\chi_I(P)E\chi_I(P)$ is compact. This concludes the assertion. \square

Proof of Theorem 4. Since I is an interval, we can take $\nu \in \mathbb{R}$ and $\tau \geq 0$ such that $I = (\nu - \tau, \nu + \tau)$. Let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi(x) = 1$ if $x \in I' = (\nu - \tau', \nu + \tau') \Subset I$ and $\text{supp } \chi \Subset I$. Then it suffices to prove Theorem 4 with χ_I is replaced by such χ . For the technical convince, we assume χ satisfies this condition with I' and τ' are replaced by I and τ .

From Lemma 5, it is sufficient to prove that

$$\chi(P)\{P_0 + \frac{1}{\lambda} \sum_{j,k} j(r)(\partial_{\theta_j} \tilde{V})h^{jk}(\theta)(\partial_{\theta_k} \tilde{V})\}\chi(P) \geq \alpha\chi(P) + K.$$

Since $I \Subset \mathbb{R} \setminus (\sigma_{pp}(P) \cup \text{Cv}(\tilde{V}))$, we can find $\eta > 0$ such that

$$I' = (\nu - \tau - \eta, \nu + \tau + \eta) \Subset \mathbb{R} \setminus (\sigma_{pp}(P) \cup \text{Cv}(\tilde{V})).$$

We define $\mathcal{O}_i \subset \partial M$ ($i = 1, 2, 3$) by

$$\begin{aligned} \mathcal{O}_1 &= \tilde{V}^{-1}(\nu - \tau - \eta, \nu + \tau + \eta), \\ \mathcal{O}_2 &= \tilde{V}^{-1}(-\infty, \nu - \tau - \frac{1}{2}\eta), \\ \mathcal{O}_3 &= \tilde{V}^{-1}(\nu + \tau + \frac{1}{2}\eta, \infty) \end{aligned}$$

Since ∂M is compact and smooth, we can take a partition of unity $\{\tilde{f}_i\}_{i=1,2,3}$ compatible with $\{\mathcal{O}_i\}_{i=1,2,3}$ i.e. there exist $\tilde{f}_i \in C^\infty(\partial M)$ such that $\text{supp}(\tilde{f}_i) \subset \mathcal{O}_i$ for $i = 1, 2, 3$ and $\sum_{i=1}^3 \tilde{f}_i^2 = 1$. We define $f_i \in C^\infty(M)$ by $f_i(r, \theta) = j(r)\tilde{f}_i(\theta)$ and $\psi \in C_c^\infty(M)$ by $\psi = 1 - \sum_{i=1}^3 f_i^2$.

Then we obtain

$$\begin{aligned} &\chi(P)\{P_0 + \frac{1}{\lambda} \sum_{j,k} j(r)(\partial_{\theta_j} \tilde{V})h^{jk}(\theta)(\partial_{\theta_k} \tilde{V})\}\chi(P) \\ &= \chi(P) \sum_{i=1}^3 f_i^2 \{P_0 + \frac{1}{\lambda} \sum_{j,k} j(r)(\partial_{\theta_j} \tilde{V})h^{jk}(\theta)(\partial_{\theta_k} \tilde{V})\}\chi(P) + K_1 \\ &= \sum_{i=1}^3 \chi(P)f_i \{P_0 + \frac{1}{\lambda} \sum_{j,k} j(r)(\partial_{\theta_j} \tilde{V})h^{jk}(\theta)(\partial_{\theta_k} \tilde{V})\}f_i \chi(P) + K_2 \end{aligned}$$

where K_1 and K_2 are compact operators and we have used the Helffer-Sjöstrand formula from Section 14 of [12] in the third line.

Since $\partial_\theta \tilde{V} \neq 0$ on $\text{supp} f_1$, there exists $\delta > 0$ such that

$$\sum_{j,k} (\partial_{\theta_j} \tilde{V})h^{jk}(\theta)(\partial_{\theta_k} \tilde{V}) > \delta$$

Then we obtain,

$$\begin{aligned} &\chi(P)f_1 \{P_0 + \frac{1}{\lambda} \sum_{j,k} j(r)(\partial_{\theta_j} \tilde{V})h^{jk}(\theta)(\partial_{\theta_k} \tilde{V})\}f_1 \chi(P) \\ &\geq \frac{\delta}{\lambda} \chi(P)f_1^2 \chi(P) + K_3 \end{aligned}$$

where K_3 is a compact operator.

Since $\nu - \tilde{V} \geq \tau + \frac{\eta}{2}$ on $\text{supp} f_2$ and $\chi(P)(P - \nu)\chi(P) \geq \tau\chi(P)^2$, we obtain the following

$$\begin{aligned}
& \chi(P)f_2\{P_0 + \frac{1}{\lambda} \sum_{j,k} j(r)(\partial_{\theta_j} \tilde{V})h^{jk}(\theta)(\partial_{\theta_k} \tilde{V})\}f_2\chi(P) \\
& \geq \chi(P)f_2P_0f_2\chi(P) + K_4 \\
& \geq \chi(P)f_2(P - \nu + \nu - \tilde{V})f_2\chi(P) + K_4 \\
& \geq \chi(P)f_2(P - \nu + \tau + \frac{\eta}{2})f_2\chi(P) + K_4 \\
& = f_2\chi(P)(P - \nu + \tau + \frac{\eta}{2})\chi(P)f_2 + K_5 \\
& \geq f_2\chi(P)(-\tau + \tau + \frac{\eta}{2})\chi(P)f_2 + K_5 \\
& = \frac{\eta}{2}\chi(P)f_2^2\chi(P) + K_6
\end{aligned}$$

where K_4 , K_5 and K_6 are compact operators.

Since $\nu - \tilde{V} \leq -\tau - \frac{\eta}{2}$ on $\text{supp} f_3$ and $\chi(P)(P - \nu)\chi(P) \leq \tau\chi(P)^2$, we obtain the following

$$\begin{aligned}
& \chi(P)f_3P_0f_3\chi(P) \\
& = \chi(P)f_3(P - \nu + \nu - \tilde{V})f_3\chi(P) \\
& \leq \chi(P)f_3(P - \nu - \tau - \frac{\eta}{2})f_3\chi(P) \\
& = f_3\chi(P)(P - \nu - \tau - \frac{\eta}{2})\chi(P)f_3 + K_7 \\
& \leq f_3\chi(P)(\tau - \tau - \frac{\eta}{2})\chi(P)f_3 + K_7 \\
& = -\frac{\eta}{2}\chi(P)f_3^2\chi(P) + K_8,
\end{aligned}$$

where K_7 and K_8 are compact operators. Thus we obtain,

$$0 \leq \chi(P)f_3(P_0 + \frac{\eta}{2})f_3\chi(P) \leq K_8$$

and hence $\chi(P)f_3(P_0 + \frac{\eta}{2})f_3\chi(P)$ is a compact operator. Let $\gamma < \frac{\eta}{2}$, then we see

$$\begin{aligned}
& \gamma\chi(P)f_3^2\chi(P) - \chi(P)f_3(P_0 + \frac{\eta}{2})f_3\chi(P) \\
& = -\chi(P)f_3P_0f_3\chi(P) - (\frac{\eta}{2} - \gamma)\chi(P)f_3^2\chi(P) \\
& \leq 0 \\
& \leq \chi(P)f_3\{P_0 + \frac{1}{\lambda} \sum_{j,k} j(r)(\partial_{\theta_j} \tilde{V})h^{jk}(\theta)(\partial_{\theta_k} \tilde{V})\}f_3\chi(P).
\end{aligned}$$

Since $\chi(P)f_3(P_0 + \frac{\eta}{2})f_3\chi(P)$ is a compact operator, we get inequality for f_3 .

Finally, by taking $\alpha = \min\{\frac{\delta}{\lambda}, \frac{\eta}{2}, \gamma\}$, we conclude the assertion. \square

Proof of Theorem 2. From Theorem 4 and the Mourre theory in [11], we only have to prove the following:

1. P is $C^2(A_\lambda)$ class i.e. $[(P-z)^{-1}, A_\lambda], A_\lambda]$ is bounded for any $z \in \rho(P)$.
2. $\langle r \rangle^{-s}(P-E \pm i0)^{-1}\langle r \rangle^{-s} = \lim_{\varepsilon \rightarrow +0} \langle r \rangle^{-s}(P-E \pm i\varepsilon)^{-1}\langle r \rangle^{-s}$ converge uniformly in $E \in I$.

We see, from (2.10) in the proof of lemma 7, $[P, A_\lambda]$ is P -bounded. We can calculate $[[P, A_\lambda], A_\lambda]$ is P -bounded similarly. Since

$$\begin{aligned} & [(P-z)^{-1}, A_\lambda], A_\lambda] \\ &= (P-z)^{-1}[P, A_\lambda][(P-z)^{-1}, A_\lambda] + (P-z)^{-1}[[P, A_\lambda], A_\lambda](P-z)^{-1} \\ &+ [(P-z)^{-1}, A_\lambda][P, A_\lambda](P-z)^{-1}, \end{aligned}$$

we learn $[[P, A_\lambda], A_\lambda]$ is bounded.

Concerning claim 2, we assume the following lemma for the moment.

Lemma 8. $\langle A_\lambda \rangle^s(P \pm i)^{-1}\langle r \rangle^{-s}$ are bounded operators on \mathcal{H} .

When $\text{Im}z \geq 1$, claim 2 is obvious and hence we may assume $\text{Im}z \leq 1$. Then we can calculate

$$\begin{aligned} & (P-z)^{-1} \\ &= (P+i)^{-1} + (z+i)(P+i)^{-2} + (z+i)^2(P+i)^{-1}(P-z)^{-1}(P+i)^{-1} \end{aligned}$$

Since first and second term in the right hand side are uniformly bounded if $\text{Im}z \leq 1$, we only have to treat the third term. Concerning the third term, we see

$$\begin{aligned} & \langle r \rangle^{-s}(z+i)^2(P+i)^{-1}(P-z)^{-1}(P+i)^{-1}\langle r \rangle^{-s} \\ &= (z+i)^2\langle r \rangle^{-s}(P+i)^{-1}\langle A_\lambda \rangle^s \\ &\quad \times \langle A_\lambda \rangle^{-s}(P-z)^{-1}\langle A_\lambda \rangle^{-s} \\ &\quad \times \langle A_\lambda \rangle^s(P+i)^{-1}\langle r \rangle^{-s}. \end{aligned}$$

From Theorem 2, we learn $\langle A_\lambda \rangle^{-s}(P-z)^{-1}\langle A_\lambda \rangle^{-s}$ is uniformly bounded and thus right hand side of the above equality is uniformly bounded and this completes the proof of Theorem 2. \square

Proof of Lemma 8. From the usual interpolation argument, it suffices to prove Lemma 8 when $s = 0, 1$. When $s = 0$, it is obvious. When $s = 1$, we see

$$A_\lambda(P \pm i)^{-1}\langle r \rangle^{-1} = (P \pm i)^{-1}[P, A_\lambda](P \pm i)^{-1}\langle r \rangle^{-1} + (P \pm i)^{-1}A_\lambda r^{-1}.$$

Proof of lemma 5 shows the right hand side of the above equality is bounded. Thus $A_\lambda(P \pm i)^{-1}\langle r \rangle^{-1}$ is also bounded, which concludes the proof. \square

2.3 Properties of $(PJ - JP_f)$.

We now prove some properties of $(PJ - JP_f)$.

Proposition 9. (1) $PJ - JP_f$ can be written as following,

$$(PJ - JP_f) = -(\tilde{J}\partial_r + \partial_r\tilde{J})\varphi + \frac{(n-1)^2 - 2(n-1)}{4r^2}J\varphi + \frac{n-1}{2r}\tilde{J}\varphi \\ - \frac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G \begin{pmatrix} a_1 - 1 & a_2 \\ {}^t a_2 & a_3 - r^2\tilde{j}h \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta/r \end{pmatrix} J + V_s J,$$

where $\tilde{J} : \mathcal{H}_f \rightarrow \mathcal{H}$ is such that

$$(\tilde{J}\varphi)(r, \theta) = r^{-(n-1)/2} (\partial_r j(r)) \varphi(r, \theta) \quad \text{if } (r, \theta) \in M_\infty,$$

and

$$\tilde{J}\varphi(x) = 0 \text{ if } x \notin M_\infty.$$

(2) $(P - z)^{-1}\langle r \rangle^s (PJ - JP_f)\langle r \rangle^s$ is bounded for some $1/2 < s < 1$ and any $z \in \rho(P)$.

(3) $\langle r \rangle^s \chi(P)(PJ - JP_f)\langle r \rangle^s$ is bounded for some $1/2 < s < 1$ and any $\chi \in C_c^\infty(\mathbb{R})$.

Proof. From direct computations, for $\varphi \in C_c^\infty(M_f)$, we learn

$$\begin{aligned} & J\partial_r^2\varphi \\ &= r^{-(n-1)/2} j(r)\partial_r^2\varphi \\ &= \partial_r r^{-(n-1)/2} j(r)\partial_r\varphi \\ &+ \{-r^{-(n-1)/2}(\partial_r j(r)) + \frac{n-1}{2r} \times r^{-(n-1)/2} j(r)\}\partial_r\varphi \\ &= \partial_r^2 r^{-(n-1)/2} j(r)\varphi \\ &+ \{-r^{-(n-1)/2}(\partial_r j(r)) + \frac{n-1}{2r} \times r^{-(n-1)/2} j(r)\}\partial_r\varphi \\ &+ \partial_r \{-r^{-(n-1)/2}(\partial_r j(r)) + \frac{n-1}{2r} \times r^{-(n-1)/2} j(r)\}\varphi \\ &= r^{-(n-1)}\partial_r r^{n-1}\partial_r r^{-(n-1)/2} j(r)\partial_r\varphi \\ &- \{r^{-(n-1)/2}(\partial_r j(r))\partial_r + \partial_r r^{-(n-1)/2}(\partial_r j(r))\}\varphi \\ &+ \frac{(n-1)^2 - 2(n-1)}{4r^2} \times r^{-(n-1)/2} j(r)\varphi \\ &+ \frac{n-1}{2r} \times r^{-(n-1)/2}(\partial_r j(r))\varphi. \end{aligned}$$

Thus we obtain,

$$\begin{aligned}
& (PJ - JP_f)\varphi \\
&= -\{r^{-(n-1)/2}(\partial_r j(r))\partial_r + \partial_r r^{-(n-1)/2}(\partial_r j(r))\}\varphi \\
&+ \frac{(n-1)^2 - 2(n-1)}{4r^2} \times r^{-(n-1)/2} j(r)\varphi \\
&+ \frac{n-1}{2r} \times r^{-(n-1)/2}(\partial_r j(r))\varphi \\
&- \frac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G \begin{pmatrix} a_1 - 1 & a_2 \\ {}^t a_2 & a_3 - r^2 \tilde{j}h \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta/r \end{pmatrix} J + V_s J \\
&= -(\tilde{J}\partial_r + \partial_r \tilde{J})\varphi + \frac{(n-1)^2 - 2(n-1)}{4r^2} J\varphi + \frac{n-1}{2r} \tilde{J}\varphi \\
&- \frac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G \begin{pmatrix} a_1 - 1 & a_2 \\ {}^t a_2 & a_3 - r^2 \tilde{j}h \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta/r \end{pmatrix} J + V_s J,
\end{aligned}$$

Concerning (2), we note that $(z - P)^{-1}\langle r \rangle^s \{ \frac{(n-1)^2 - 2(n-1)}{4r^2} J + \frac{n-1}{2r} \tilde{J} \} \langle r \rangle^{-s}$ is bounded as $\langle r \rangle^s \{ \frac{(n-1)^2 - 2(n-1)}{4r^2} J + \frac{n-1}{2r} \tilde{J} \}$ is bounded.

Also, $(z - P)^{-1}\langle r \rangle^s \partial_r \tilde{J} \langle r \rangle^s$ is bounded since ∂_r is P -bounded and $\langle r \rangle^{2s} \tilde{J}$ is bounded for any s .

For $\varphi \in C_c^\infty(M_f)$, we see

$$\begin{aligned}
& \tilde{J}\partial_r \varphi \\
&= \partial_r \tilde{J}\varphi + \{-r^{-(n-1)/2}(\partial_r^2 j(r)) + \frac{n-1}{2r} r^{-(n-1)/2}(\partial_r j(r))\}\varphi.
\end{aligned}$$

Thus we obtain $[\tilde{J}, \partial_r] = -r^{-(n-1)/2}\partial_r^2 j(r) + \frac{n-1}{2r} r^{-(n-1)/2}\partial_r j(r)$ and so $(z - P)^{-1}\langle r \rangle^s [\tilde{J}, \partial_r] \tilde{J} \langle r \rangle^s$ is bounded. To sum up, $(z - P)^{-1}\langle r \rangle^s \tilde{J}\partial_r \langle r \rangle^s = (z - P)^{-1}\langle r \rangle^s [\tilde{J}, \partial_r] \tilde{J} \langle r \rangle^s + (z - P)^{-1}\langle r \rangle^s \partial_r \tilde{J} \langle r \rangle^s$ is bounded. From these calculation and definition of \tilde{j} and the assumption for P , we obtain (2).

Concerning (3), first we calculate as following,

$$\begin{aligned}
& \langle r \rangle^s \chi(P)(PJ - JP_f)\langle r \rangle^s \\
&= \chi(P)\langle r \rangle^s (PJ - JP_f)\langle r \rangle^s + [\langle r \rangle^s, \chi(P)](PJ - JP_f)\langle r \rangle^s \\
&= \chi(P)(P - i)(P - i)^{-1}\langle r \rangle^s (PJ - JP_f)\langle r \rangle^s + [\langle r \rangle^s, \chi(P)](PJ - JP_f)\langle r \rangle^s.
\end{aligned}$$

First term is bounded from (2). Thus the problem is the second term.

From the Helffer-Sjöstrand formula, we obtain

$$\begin{aligned}
& [\langle r \rangle^s, \chi(P)](PJ - JP_f)\langle r \rangle^s \\
&= (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) [\langle r \rangle^s, (z - P)^{-1}] dz \wedge d\bar{z} (PJ - JP_f)\langle r \rangle^s \\
&= (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - P)^{-1} [\langle r \rangle^s, P] (z - P)^{-1} dz \wedge d\bar{z} (PJ - JP_f)\langle r \rangle^s \\
&= (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - P)^{-1} [\langle r \rangle^s, P] (z - P)^{-1} (P - i) dz \wedge d\bar{z} \\
&\quad \times (P - i)^{-1} (PJ - JP_f)\langle r \rangle^s
\end{aligned}$$

Similarly to the proof of (2), we can see $(P - i)^{-1}(PJ - JP_f)\langle r \rangle^s$ is bounded. Concerning the integrability of above integral, we only have to prove the integrability in $\text{Im}z < 1$. That is because $\tilde{\chi}$ is the almost analytic extension and $(P - z)^{-1}$ has no critical point in $\text{Im}z > 1$. Since $\|(z - P)^{-1}(i - P)\| = O(\text{Im}z^{-1})$ and $\|(z - P)^{-1}\| = O(\text{Im}z^{-1})$ if $\text{Im}z < 1$, the integrand is bounded uniformly in z , where we have used the fact $|\partial_{\bar{z}} \tilde{\chi}(z)| = O((\text{Im}z)^2)$ as it is almost analytic extension. Thus we obtain Proposition 9. \square

3 Existence and completeness of W_{\pm}

We prepare some lemmas to prove Theorem 1.

Lemma 10. *Let $\chi \in C_c^\infty(\mathbb{R})$ then $\chi(P)J - J\chi(P_f)$ is a compact operator.*

Proof. From the Helffer-Sjöstrand formula, we see

$$\begin{aligned}
& \chi(P)J - J\chi(P_f) \\
&= (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) \{(z - P)^{-1}J - J(z - P_f)^{-1}\} dz \wedge d\bar{z} \\
&= -(2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - P)^{-1} (PJ - JP_f) (z - P_f)^{-1} dz \wedge d\bar{z}
\end{aligned}$$

From Proposition 9, it is sufficient to prove the following claim.

- Let T be an operator from \mathcal{H}_f to \mathcal{H} . We assume $(z - P)^{-1}\langle r \rangle^s T \langle r \rangle^s$ is bounded for some $1/2 < s < 1$ and any $z \in \mathbb{C}$. Then one can see that $(z - P)^{-1}T(z - P_f)^{-1}$ is a compact operator.

We can calculate as following,

$$\begin{aligned}
& (z - P)^{-1}T(z - P_f)^{-1} \\
&= \langle r \rangle^{-s} (z - P)^{-1} \langle r \rangle^s T \langle r \rangle^s \times \langle r \rangle^{-s} (z - P_f)^{-1} \\
&\quad + \langle r \rangle^{-s} (z - P)^{-1} [P, \langle r \rangle^s] (z - P)^{-1} T \langle r \rangle^s \langle r \rangle^{-s} (z - P_f)^{-1}.
\end{aligned}$$

From the assumption for the claim and the locally compactness of P_f , first term is compact. Since $s < 1$ implies $[P, \langle r \rangle^s]$ is P -bounded and the fact $(z-P)^{-1}T\langle r \rangle^s$ is bounded can be proved similarly to the proof of Proposition 7, second term is also compact and we conclude the proof of the claim.

We apply this claim with $T = (PJ - JP_f)$ to obtain Lemma 10. \square

Lemma 11. *Let $\chi \in C_c^\infty(\mathbb{R})$ then $\langle r \rangle^s \chi(P) \langle r \rangle^{-s}$ is bounded operator on \mathcal{H} for any $1/2 < s < 1$.*

Proof. From the Helffer-Sjöstrand formula, we learn

$$\begin{aligned} & \langle r \rangle^s \chi(P) \langle r \rangle^{-s} \\ &= (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) \langle r \rangle^s (z - P)^{-1} \langle r \rangle^{-s} dz \wedge d\bar{z}. \end{aligned}$$

Thus it is sufficient to prove the integrability of $\partial_{\bar{z}} \tilde{\chi}(z) \langle r \rangle^s (z - P)^{-1} \langle r \rangle^{-s}$. We may assume that $\text{Im} z < 1$ since the integrability on $\text{Im} z > 1$ is obvious.

From the direct computations, we obtain

$$\begin{aligned} & \langle r \rangle^s (z - P)^{-1} \langle r \rangle^{-s} \\ &= (z - P)^{-1} + (z - P)^{-1} [P, \langle r \rangle^s] (z - P)^{-1} \langle r \rangle^{-s} \\ &= (z - P)^{-1} \\ &+ (z - P)^{-1} (i - P) \times (i - P)^{-1} [P, \langle r \rangle^s] \times (z - P)^{-1} \langle r \rangle^{-s}. \end{aligned}$$

From the fact that the order of $\|(z - P)^{-1}(i - P)\|$ and $\|(z - P)^{-1}\|$ is $(\text{Im} z)^{-1}$, and that $[P, \langle r \rangle^s]$ is P -bounded for $s \leq 1$, $\partial_{\bar{z}} \tilde{\chi}(z) \langle r \rangle^s (z - P)^{-1} \langle r \rangle^{-s}$ is integrable on $\text{Im} z < 1$ as $\tilde{\chi}(z)$ is the almost analytic extension of $\chi \in C_c^\infty(\mathbb{R})$. \square

Proof of Theorem 1. We only have to prove the existence of the wave operator for $\varphi \in \mathcal{H}_f$ such that there exists $I \in \mathbb{R} \setminus (\sigma_{pp}(P) \cup \sigma_{pp}(P_f) \cup \text{Cv}(\tilde{V}))$ which satisfies $\varphi = \chi_I(P_f)\varphi$. That is because $\sigma_{pp}(P) \cup \sigma_{pp}(P_f) \cup \text{Cv}(\tilde{V})$ is discrete from assumption and Theorem 2, and hence such φ are dense in $H_{ac}(P_f)$.

For such φ , we obtain

$$\begin{aligned} & e^{itP} J e^{-itP_f} P_{ac}(P_f) \varphi \\ &= e^{itP} \chi(P) J \chi(P_f) e^{-itP_f} P_{ac}(P_f) \varphi + e^{itP} (J \chi(P_f) - \chi(P) J) e^{-itP_f} P_{ac}(P_f) \varphi \end{aligned}$$

From Lemma 10, $(J \chi(P_f) - \chi(P) J)$ is compact. Thus we obtain

$$\lim_{t \rightarrow \pm\infty} \|e^{itP} (J \chi(P_f) - \chi(P) J) e^{-itP_f} P_{ac}(P_f) \varphi\| = 0.$$

Then it is sufficient to prove $\lim_{t \rightarrow \pm\infty} e^{itP} \chi(P) J \chi(P_f) e^{-itP_f} P_{ac}(P_f) \varphi$ exist.

By differentiating $e^{itP} \chi(P) J \chi(P_f) e^{-itP_f} P_{ac}(P_f)$, we obtain the following,

$$\begin{aligned} & \frac{d}{dt} \{e^{itP} \chi(P) J \chi(P_f) e^{-itP_f} P_{ac}(P_f) \varphi\} \\ &= i e^{itP} \chi(P) (PJ - JP_f) \chi(P_f) e^{-itP_f} P_{ac}(P_f) \varphi \\ &= i e^{itP} A_1^* A_2 e^{-itP_f} P_{ac}(P_f) \varphi, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \langle r \rangle^s (P_f J^* - J^* P) \chi(P) \\ A_2 &= \langle r \rangle^{-s} \chi(P_f). \end{aligned}$$

From direct computations, we obtain

$$\begin{aligned} & A_1 (P - \lambda \pm i\varepsilon)^{-1} A_1^* \\ &= \langle r \rangle^s (P_f J^* - J^* P) \chi(P) (P - \lambda \pm i\varepsilon)^{-1} \chi(P) (PJ - JP_f) \langle r \rangle^s \\ &= \langle r \rangle^s (P_f J^* - J^* P) \chi(P) \langle r \rangle^s \\ &\times \langle r \rangle^{-s} (P - \lambda \pm i\varepsilon)^{-1} \langle r \rangle^{-s} \\ &\times \langle r \rangle^s \chi(P) (PJ - JP_f) \langle r \rangle^s. \end{aligned}$$

Proposition 9 implies first and third part are bounded. Theorem 2 yields second part is bounded uniformly for $\varepsilon > 0$. Thus A_1 is relatively P -smooth from Kato's characterization of relatively smoothness, especially

$$\frac{1}{2\pi} \sup_{\|u\|=1, u \in \mathcal{D}(P)} \int_{-\infty}^{\infty} \|A_1 e^{-itP} u\|^2 dt < \infty.$$

The fact that A_2 is P_f -smooth is proved directly from Theorem 2.

Let $\phi \in L^2(M)$, then we obtain

$$\begin{aligned}
& \left| \langle \phi, e^{itP} \chi(P) J \chi(P_f) e^{-itP_f} P_{ac}(P_f) \varphi \rangle \right| \\
&= \left| -i \int_0^t \langle \phi, e^{itP} \chi(P) (PJ - JP_f) \chi(P_f) e^{-itP_f} P_{ac}(P_f) \varphi \rangle dt \right. \\
&\quad \left. + \langle \phi, \chi(P) J \chi(P_f) P_{ac}(P_f) \varphi \rangle \right| \\
&\leq \int_{-\infty}^{\infty} \left| \langle \phi, e^{itP} \chi(P) (PJ - JP_f) \chi(P_f) e^{-itP_f} P_{ac}(P_f) \varphi \rangle \right| dt \\
&\quad + \left| \langle \phi, \chi(P) J \chi(P_f) P_{ac}(P_f) \varphi \rangle \right| \\
&= \int_{-\infty}^{\infty} \left| \langle A_1 e^{-itP} \phi, A_2 e^{-itP_f} P_{ac}(P_f) \varphi \rangle \right| dt + \left| \langle \phi, \chi(P) J \chi(P_f) P_{ac}(P_f) \varphi \rangle \right| \\
&\leq \int_{-\infty}^{\infty} \|A_1 e^{-itP} \phi\| \|A_2 e^{-itP_f} P_{ac}(P_f) \varphi\| dt + \left| \langle \phi, \chi(P) J \chi(P_f) P_{ac}(P_f) \varphi \rangle \right| \\
&\leq \left(\int_{-\infty}^{\infty} \|A_1 e^{-itP} \phi\|^2 dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \|A_2 e^{-itP_f} P_{ac}(P_f) \varphi\|^2 dt \right)^{\frac{1}{2}} \\
&\quad + \left| \langle \phi, \chi(P) J \chi(P_f) P_{ac}(P_f) \varphi \rangle \right|.
\end{aligned}$$

Thus $\lim_{t \rightarrow \pm\infty} e^{itP} \chi(P) J \chi(P_f) e^{-itP_f} P_{ac}(P_f) \varphi$ exist and hence W_{\pm} exist.

We can prove the existence of $\tilde{W}_{\pm} = \text{s-lim}_{t \rightarrow \pm\infty} e^{itP_f} J^* e^{-itP} P_{ac}(P)$ as follows:

- Concerning J^* , counterpart of Proposition 9 is proved similarly by exchanging P by P_f and J replaced with J^* .
- Then Lemma 10 is also proved similarly.
- Lemma 11 is directly proved by P replaced by P_f .
- It is sufficient to prove $\lim_{t \rightarrow \pm\infty} e^{itP_f} \chi(P_f) J^* \chi(P) e^{-itP} P_{ac}(P) \varphi$ exist to prove \tilde{W} exist, where $\varphi \in \mathcal{H}_I$ is such that $\varphi = \chi_I(P_f) \varphi$ with $I \subseteq \mathbb{R} \setminus (\sigma_{pp}(P) \cup \sigma_{pp}(P_f) \cup \text{Cv}(\tilde{V}))$.
- Relatively smoothness argument in the proof of Theorem 1 can be recovered by replacing A_1 and A_2 by A_2^* and A_1^* .

□

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